TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 267, Number 2, October 1981

SUBSPACES OF L^1, VIA RANDOM MEASURES

BY
DAVID J. ALDOUS[1]

ABSTRACT. It is shown that every subspace of L^1 contains a subspace isomorphic to some l_q. The proof depends on a fixed point theorem for random measures.

1. Introduction. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let L^1 be the Banach space of integrable random variables. By a *subspace* of L^1 we mean a closed infinite-dimensional linear subspace. This paper is devoted to the proof of the following theorem: for the general background to this result and the theory of L^p spaces see [7], [8], [11].

THEOREM 1.1. *Any subspace H of L^1 contains a subspace isomorphic to l_q, for some $q \in [1, 2]$.*

By a result of Rosenthal [10] we need consider only subspaces satisfying

$$\text{the unit ball of } H \text{ is uniformly integrable,} \tag{1.2}$$

for any H failing (1.2) contains a subspace isomorphic to l_1. Except for this observation and Proposition 3.15, our proof is essentially self-contained.

Dacunha-Castelle and Krivine [5] investigated subspaces of L^1 using ultraproduct techniques, and showed that it would suffice to prove Theorem 1.1 for subspaces generated by exchangeable sequences (see also Maurey and Schectmann [9]). We eschew ultraproducts in favor of random measures, which the author finds more comprehensible. An account of random measures from a functional analysis viewpoint is given in §2.

There is a natural correspondence between exchangeable sequences and random measures. The only real novelty in our approach is that we regard the random measures, rather than exchangeable sequences, as the objects to study. From any sequence of random variables we may extract a subsequence which somewhat resembles an exchangeable sequence. So let $\mathcal{C}$ be the class of random measures corresponding to exchangeable sequences arising in this way from a given subspace H of L^1. On the one hand, $\mathcal{C}$ must satisfy certain structural properties (Proposition 3.9). On the other hand, if $\mathcal{C}$ contains a random measure of a certain special kind (a randomly scaled symmetric stable law of exponent q) then H contains l_q (Proposition 3.11). Thus Theorem 1.1 is reduced to a result involving only random

---

Received by the editors May 8, 1980 and, in revised form, September 26, 1980.
1980 *Mathematics Subject Classification.* Primary 46B05, Secondary 60G55.
Key words and phrases. Banach space, L^p space, random measure, exchangeable sequences, symmetric stable laws, fixed point theorem.
[1]This research was supported in part by National Science Foundation Grant MCS78-25301. Paper presented at International Conference on Banach Spaces, Kent State University, August 1979.

0002-9947/81/0000-0455/$05.75

measures (Theorem 3.10). This result can be regarded as a fixed point theorem, but known general results do not seem applicable: our ad hoc proof occupies §§4 and 5. §6 contains a miscellany of remarks and conjectures.

REMARK ADDED SEPTEMBER 1980. Since the first draft of this paper, much progress has been made in this field. Krivine and Maurey have generalised Theorem 1.1 to show that any Banach space which is *stable* (in the sense of **[12]**) contains an isomorphic copy of some l_p. Their result was announced in **[12]**; for more details see **[13]**. In unpublished lecture notes, Garling has given a fine synthesis of the present paper and the work of Krivine and Maurey; he also shows that certain Orlicz function spaces are stable.

Notation. For random variables (r.v.'s) $X_1, X_2, \ldots$ write $X_n \underset{p}{\to} X$ for convergence in probability, and write $X_n \overset{s}{\to} X$ (resp. $\overset{w}{\to} X$) for strong (resp. weak) convergence in L^1. Let $\mathcal{P}$ denote the set of probability measures on $\mathbf{R}$. Let $\mathcal{L}(X) \in \mathcal{P}$ denote the law of a r.v. X. Write $\langle f, \mu \rangle$ for $\int f\, d\mu$. Give $\mathcal{P}$ the usual topology: $\mu_n \to \mu$ iff $\langle f, \mu_n \rangle \to \langle f, \mu \rangle$ for each $f \in C(\mathbf{R})$, where $C(\mathbf{R})$ is the set of bounded continuous functions. Recall that $\Lambda \subset \mathcal{P}$ is relatively compact iff it is *tight*, that is

$$\inf\{\lambda([-n, n]) : \lambda \in \Lambda\} \to 1 \quad \text{as } n \to \infty.$$

For $\lambda \in \mathcal{P}$ write $\phi_\lambda(t)$ for the Fourier transform $\int e^{itx}\lambda\,(dx)$, and write $|\lambda| = \int |x| \lambda\,(dx)$. Let $\delta_a \in \mathcal{P}$ denote the measure degenerate at a.

2. Random measures. By random measure we mean a random probability measure. Thus a random measure is a measurable function $\xi: \Omega \to \mathcal{P}$, in other words a $\mathcal{P}$-valued random variable. Let $\mathfrak{M}$ denote the set of random measures. For $\xi \in \mathfrak{M}$ and f bounded measurable, the expression $\langle f, \xi \rangle$ defines a random variable $\omega \to \langle f, \xi(\omega) \rangle$. For $A \subset \Omega$ with $P(A) > 0$ define $\xi^A \in \mathcal{P}$ by

$$\xi^A(\cdot) = [P(A)]^{-1} \int_A \xi(\omega, \cdot) P\,(d\omega),$$

and think of ξ^A as the average of ξ over A. Call ξ *constant* if $\xi \equiv \xi_0$ for some $\xi_0 \in \mathcal{P}$; *degenerate* if $\xi = \delta_\alpha$ for some r.v. α.

EXAMPLE 2.1. For $1 < q \leqslant 2$ and $a \geqslant 0$ let $\sigma(q, a) \in \mathcal{P}$ denote the measure with $\phi(t) = \exp(-a^q|t|^q)$. That is, $\sigma(q, a)$ is a symmetric stable law of exponent q, where a can be regarded as a scale parameter. Now for a r.v. $V \geqslant 0$ we can define a random measure $\xi = \sigma(q, V)$. If V has law ν, write $\sigma(q, \nu)$ for ξ^Ω. To look at this from a probabilistic viewpoint, let S be independent of V with law $\sigma(q, 1)$. Then $\sigma(q, \nu)$ is the law of VS, and $\sigma(q, V)$ is the conditional law of VS given V.

Random measures of this form play a large part in the sequel.

Many topologies can be defined on $\mathfrak{M}$, but we shall be concerned only with the next two. Define

$$\xi_n \overset{sm}{\to} \xi \quad \text{iff} \quad \langle f, \xi_n \rangle \overset{s}{\to} \langle f, \xi \rangle, \qquad f \in C(\mathbf{R}), \tag{2.2}$$

$$\xi_n \overset{wm}{\to} \xi \quad \text{iff} \quad \langle f, \xi_n \rangle \overset{w}{\to} \langle f, \xi \rangle, \qquad f \in C(\mathbf{R}). \tag{2.3}$$

These definitions can be reformulated in several ways. Clearly

$$\xi_n \overset{wm}{\to} \xi \quad \text{iff} \quad \xi_n^A \to \xi^A, \qquad P(A) > 0. \tag{2.4}$$

And it is easy to verify that *sm*-convergence coincides with the usual notion of convergence in probability for metric space valued random variables:

$$\xi_n \xrightarrow{sm} \xi \quad \text{iff} \quad d(\xi_n, \xi) \underset{p}{\rightarrow} 0 \tag{2.5}$$

where d is any metrisation of $\mathcal{P}$. When this holds, we can find a subsequence such that $\xi_{n_j} \rightarrow \xi$ a.s., and this gives a useful technique for proving facts about *sm*-convergence.

Other reformulations can be given in terms of transforms, and these will be needed when we consider convolutions. For $\xi \in \mathfrak{M}$ let

$$\phi_\xi(t, \omega) = \phi_{\xi(\omega)}(t) = \langle \exp(it\,\cdot), \xi(\omega, \cdot) \rangle$$

be the random Fourier transform. Unsurprisingly,

$$\xi_n \xrightarrow{sm} \xi \quad \text{iff} \quad \phi_{\xi_n}(t) \xrightarrow{s} \phi_\xi(t) \quad \text{for all } t, \tag{2.6}$$

$$\xi_n \xrightarrow{wm} \xi \quad \text{iff} \quad \phi_{\xi_n}(t) \xrightarrow{w} \phi_\xi(t) \quad \text{for all } t, \tag{2.7}$$

and it suffices to consider t in a dense set containing 0.

Let $*$ denote convolution in $\mathcal{P}$. Then $*$ is continuous on $\mathcal{P}$. For $\xi_1, \xi_2 \in \mathfrak{M}$ let $\xi_1 * \xi_2$ denote the pointwise convolution $\xi_1(\omega) * \xi_2(\omega)$. It is clear from (2.6) that $*$ is *sm*-continuous. But $*$ is not *wm*-continuous, as the next example shows.

EXAMPLE 2.8. With the notation of Example 2.1, let $V_1, V_2, \ldots$ be independent with law ν, and define $\xi_i = \sigma(q, V_i)$. Then

$$\xi_i \xrightarrow{wm} \sigma(q, \nu),$$

$$\xi_i * \xi_{i+1} \xrightarrow{wm} \sigma(q, \lambda) = \sigma(q, \nu) * \sigma(q, \nu),$$

$$\xi_i * \xi_i \xrightarrow{wm} \sigma(q, \mu),$$

where λ is the law of $(V_1^q + V_2^q)^{1/q}$, μ is the law of $2^{1/q}V_1$, and $\lambda \neq \mu$ in general.

The usefulness of the *wm* topology lies in its compactness properties. Call $\mathfrak{M}_0 \subset \mathfrak{M}$ *tight* if $\{\xi^\Omega : \xi \in \mathfrak{M}_0\}$ is tight, that is if

$$\inf_{\mathfrak{M}_0} E\langle 1_{[-n,n]}, \xi \rangle \rightarrow 1 \quad \text{as } n \rightarrow \infty. \tag{2.9}$$

In particular, $\mathfrak{M}_0$ is tight if $\sup\{E|\xi| : \xi \in \mathfrak{M}_0\} < \infty$.

LEMMA 2.10. *A tight subset of $\mathfrak{M}$ is relatively wm-compact.*

PROOF. Fix $K = [-n, n]$ and let $\mathfrak{M}_1$ be the set of random measures with support in K. We shall prove that $\mathfrak{M}_1$ is *wm*-compact, and an easy approximation argument completes the proof.

Let $\mathfrak{N}$ be the space of finite signed measures on K, and consider $\mathfrak{N}$ as the dual of $C(K)$. Let $L^2(\mathfrak{N})$ and $L^2(C(K))$ be the associated spaces of square-integrable random variables. For $Z \in L^2(C(K))$, $\theta \in L^2(\mathfrak{N})$ the map $(Z, \theta) \rightarrow E\langle Z, \theta \rangle$ shows that $L^2(\mathfrak{N})$ is contained in the dual of $L^2(C(K))$. Since $\mathfrak{M}_1$ is a bounded subset of $L^2(\mathfrak{N})$, it is compact in the weak* topology, that is the topology

$$\xi_n \rightarrow \xi \quad \text{iff} \quad E\langle Z, \xi_n \rangle \rightarrow E\langle Z, \xi \rangle, \qquad Z \in L^2(C(K)). \tag{2.11}$$

By considering Z of the form $f1_A$ ($f \in C(K)$, $A \subset \Omega$), it can be seen that (2.11) coincides with the *wm*-topology.

This last argument gives another reformulation of (2.3):

$$\xi_n \overset{wm}{\to} \xi \quad \text{iff} \quad \langle Z, \xi_n \rangle \overset{w}{\to} \langle Z, \xi \rangle, \qquad Z \in L^2(C(\mathbf{R})). \tag{2.12}$$

We now give some more technical facts. The first is a general form of the continuous mapping theorem.

LEMMA 2.13. *Let* g: $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *be continuous. Define* $\hat{g}$: $\mathbf{R} \times \mathcal{P} \to \mathcal{P}$ *by* $\hat{g}(a, \mathcal{L}(X)) = \mathcal{L}(g(a, X))$. *Let* α *be an arbitrary random variable. If* $\xi_n \overset{sm}{\to} \xi$ *then* $\hat{g}(\alpha, \xi_n) \overset{sm}{\to} \hat{g}(\alpha, \xi)$; *if* $\xi_n \overset{wm}{\to} \xi$ *then* $\hat{g}(\alpha, \xi_n) \overset{wm}{\to} \hat{g}(\alpha, \xi)$.

PROOF. The *sm*-convergence case follows from (2.5) and the continuity of $\hat{g}$. The *wm*-convergence case follows from (2.12) and the identity

$$\langle f, \hat{g}(\alpha, \xi) \rangle = \langle f \circ g(\alpha, \cdot), \xi \rangle.$$

We can now show that the two topologies coincide at degenerate random measures.

LEMMA 2.14. *If* $\xi_n \overset{wm}{\to} \delta_V$ *then* $\xi_n \overset{sm}{\to} \delta_V$.

PROOF. In the special case $\xi_n \overset{wm}{\to} \delta_0$ we have $\xi_n^\Omega \to \delta_0$ in $\mathcal{P}$ by (2.4), and it is straightforward to show $\xi_n \overset{sm}{\to} \delta_0$. For the general case, let $g(a, b) = a - b$, $h(a, b) = a + b$. If $\xi_n \overset{wm}{\to} \delta_V$ then by Lemma 2.13 $\hat{g}(V, \xi_n) \overset{wm}{\to} \hat{g}(V, \delta_V) = \delta_0$. So $\hat{g}(V, \xi_n) \overset{sm}{\to} \delta_0$, and then by Lemma 2.13 $\xi_n = \hat{h}(V, \hat{g}(V, \xi_n)) \overset{sm}{\to} \hat{h}(V, \delta_0) = \delta_V$.

Finally, let L^0 denote the space of all r.v.'s, equipped with the topology of convergence in probability. There is a natural embedding i: $L^0 \to \mathfrak{M}$ given by $i(X) = \delta_X$. It is easily seen from (2.5) that $\delta_{X_n} \overset{sm}{\to} \delta_X$ implies $X_n \underset{p}{\to} X$, and so by Lemma 2.14

$$L^0 \text{ is homeomorphic to } (i(L^0), wm). \tag{2.15}$$

Moreover $i(L^0)$ can be shown to be *wm*-dense in $\mathfrak{M}$ (we do not use this fact later), and so we may regard the embedding i as producing a kind of local compactification of L^0, by Lemma 2.10.

So far, we have been discussing measures on $\mathbf{R}$, but it will also be necessary to consider measures on the compactified half-line $\mathbf{R}^+ = [0, \infty]$. Write $\mathcal{P}^+$ for the space of such probability measures, and $\mathfrak{M}^+$ for the space of random measures. The theory of $\mathfrak{M}^+$ differs only slightly from that of $\mathfrak{M}$. In particular, (2.6) and (2.7) hold for *Laplace* transforms $\phi_\alpha(t) = Ee^{-\alpha t}$, $t > 0$. Also, $\mathfrak{M}^+$ itself is *wm*-compact. We shall need one extra technical fact, an extension of Lemma 2.14 whose proof is omitted.

LEMMA 2.16. *There exists a bounded metrisation* ρ *of* $\mathcal{P}^+$ *such that* $\xi_n \overset{wm}{\to} \xi$ *in* $\mathfrak{M}^+$ *implies* $E\rho(\xi_n, \delta_\alpha) \to E\rho(\xi, \delta_\alpha)$ *for any r.v.* $\alpha \geqslant 0$.

REMARK. Recall that a sequence $Z_1, Z_2, \ldots,$ of random variables is called exchangeable with canonical random measure ξ if, conditional on $\xi = \lambda$, the random variables (Z_i) are independent with common law λ. Our proof of Theorem

1.1 does not actually use exchangeable sequences. However, some of our manipulations with random measures are motivated by considering the corresponding manipulations of exchangeable sequences, so from time to time we provide translations between the two languages. If $\xi = \sigma(q, \alpha)$ then span(Z_i) is isomorphic to l_q: hence our interest in these special random measures.

REMARK. Properties of the *wm*-topology have been presented above from the viewpoint of functional analysis, but these ideas arise naturally in a probabilistic context [2]. Given r.v.'s (X_n) and $\mu \in \mathcal{P}$, write $X_n \Rightarrow \mu$ for convergence in law, i.e., $\mathcal{L}(X_n) \to \mu$ in $\mathcal{P}$. By requiring that convergence to μ should still hold conditional on any fixed event, we obtain the idea of *mixing*:

$$X_n \Rightarrow \mu \text{ (mixing)} \quad \text{if } E(f(X_n)|A) \to \langle f, \mu\rangle, f \in C(\mathbf{R}), P(A) > 0.$$

Clearly this is equivalent to $i(X_n) \overset{wm}{\to} \mu$. By allowing the limiting law to depend on the conditioning event, we obtain the idea of *stability*:

$$X_n \Rightarrow \mu \text{ (stably) if for each } A \text{ with } P(A) > 0 \text{ there exists } \mu_A$$
$$\text{such that } E(f(X_n)|A) \to \langle f, \mu_A\rangle, f \in C(\mathbf{R}).$$

It can be shown that X_n converges (stably) if and only if $i(X_n) \overset{wm}{\to} \xi$ for some ξ, and then $\mu_A = \xi^A$. Thus one could say that X_n converges (stably) to the random measure ξ. Lemma 2.10 shows that from any tight (X_n) we can extract a subsequence converging stably to some ξ. This is the starting point of [1], where it is shown that we can make the subsequence similar (in certain senses) to the exchangeable sequence corresponding to ξ. The present paper proceeds in a different direction: we shall consider the set of ξ which can be obtained from sequences in the given subspace H of L^1.

REMARK. The major development of the theory of random measures has been in a completely different area, as models for point processes–see [6]. There the main interest lies in the distributional properties, which do not concern us here.

3. l_q and a class of random measures. We now introduce certain classes of random measures, to be called C-classes. The idea is to abstract the properties of the *wm*-closure of a subspace of L^1 (under the natural embedding). Essentially, a C-class is a class closed under the operations of scaling and convolution, and closed in the *wm*-topology. It turns out that we can also impose some integrability and symmetry conditions. Formally, let $\mathcal{P}_0 \subset \mathcal{P}$ be the set of symmetric integrable laws. For $0 \leqslant a < \infty$, let s_a: $\mathcal{P} \to \mathcal{P}$ denote scaling by the factor a, so that $s_a(\mathcal{L}(V)) = \mathcal{L}(aV)$. Clearly

$$i(X_n) \overset{wm}{\to} \xi \text{ implies } i(aX_n) \overset{wm}{\to} s_a(\xi). \tag{3.1}$$

DEFINITION. *A class $\mathcal{C}$ of random measures is a C-class if*

$$\xi(\omega) \in \mathcal{P}_0 \text{ a.s., } \quad \text{each } \xi \in C, \tag{3.2}$$

$$E|\xi| < \infty, \quad \text{each } \xi \in C, \tag{3.3}$$

$$\mathcal{C} \text{ is wm-closed,} \tag{3.4}$$

$$\text{if } \xi \in \mathcal{C} \text{ then } s_a(\xi) \in \mathcal{C}, \qquad 0 \leqslant a < \infty, \tag{3.5}$$

$$\textit{if } \xi_1, \xi_2 \in \mathcal{C} \textit{ then } \xi_1 * \xi_2 \in \mathcal{C}, \tag{3.6}$$

$$\textit{if } \xi_n \in \mathcal{C} \textit{ and } \xi_n \overset{wm}{\to} \xi \textit{ then } E|\xi_n| \to E|\xi|. \tag{3.7}$$

Every nonempty C-class contains δ_0, by (3.5): call a class *nontrivial* if it contains some other element. Using Lemma 2.10 we obtain

$$\{\xi \in \mathcal{C}: E|\xi| \leqslant K\} \text{ is } wm\text{-compact.} \tag{3.8}$$

EXAMPLE. For $q \in (1, 2]$ and a fixed r.v. $\alpha \geqslant 0$ with $E\alpha < \infty$, the class $\{\sigma(q, c\alpha): c \geqslant 0\}$ is a C-class. However, the class $\{\sigma(q, \beta): \beta \geqslant 0, \beta \text{ integrable}\}$ is not a C-class, because by Example 2.8 it is not wm-closed.

Theorem 1.1 can now be decomposed into the three assertions below, which together establish the theorem.

PROPOSITION 3.9. *Let H be a subspace of L^1 satisfying* (1.2). *Let $\mathcal{C}[H]$ be the set of ξ in the wm-closure of $i(H)$ such that $\xi(\omega) \in \mathcal{P}_0$ a.s. Then $\mathcal{C}[H]$ is a nontrivial C-class. Moreover, if $X_n \in H$ and $i(X_n) \overset{wm}{\to} \xi$ then (X_n) is uniformly integrable.*

THEOREM 3.10. *Let $\mathcal{C}$ be a nontrivial C-class. Then $\mathcal{C}$ contains $\sigma(q, \alpha)$ for some $q \in (1, 2]$ and some $\alpha \geqslant 0$ with $P(\alpha > 0) > 0$.*

PROPOSITION 3.11. *Let (X_n) be a uniformly integrable sequence of r.v.'s such that $i(X_n) \overset{wm}{\to} \sigma(q, \alpha)$ for some $\alpha \geqslant 0$ with $P(\alpha > 0) > 0$. Then some subsequence $(W_n) = (X_{j_n})$ is isomorphic to the unit vector basis of l_q.*

We shall prove Propositions 3.9 and 3.11 in this section, and defer Theorem 3.10 until the next section.

Before the proofs, let us try to explain what is going on in terms of subspaces F generated by exchangeable sequences (Z_i) with canonical random measures ξ. For such subspaces, when does $\hat{F}$ embed into F? Certainly it does when $\hat{Z}_i = a_1 Z_{(n-1)i+1} + \cdots + a_n Z_{ni}$ for some constants $(a_1, \ldots, a_n)$. In this case $\hat{\xi} = s_{a_1}(\xi) * \cdots * s_{a_n}(\xi)$. Let $\mathfrak{B}(\xi)$ be the set of such $\hat{\xi}$. Then the wm-closure of $\mathfrak{B}(\xi)$ is essentially a C-class, and so by Theorem 3.10 contains some $\sigma(q, \alpha)$. If we could prove that $\{\hat{\xi}: \hat{F} \text{ embeds into } F\}$ were wm-closed, it would immediately follow that l_q embeds into F. Unfortunately we are unable to prove this, because wm-convergence of random measures corresponds to a rather obscure operation on exchangeable sequences. We therefore proceed indirectly. Any exchangeable $(\hat{Z}_j)$ satisfies $i(\hat{Z}_j) \overset{wm}{\to} \hat{\xi}$. We have already seen that there are $\hat{\xi}$ in $\mathfrak{B}(\xi)$ wm-convergent to some $\sigma(q, \alpha)$, and hence there are random variables (Y_j) in F with $i(Y_j) \overset{wm}{\to} \sigma(q, \alpha)$. Proposition 3.11 now shows that l_q embeds into F. Analyzing the argument, we find the only property of F actually needed is: there exists ξ such that each $\hat{\xi}$ in $\mathfrak{B}(\xi)$ is a wm-limit of $i(Y_j)$, for some $Y_j \in F$. But this holds for a general subspace H, by Proposition 3.9.

PROOF OF PROPOSITION 3.9. Let $\mathfrak{D}$ be the wm-closure of $i(H)$. We shall show that $\mathfrak{D}$ satisfies the conditions for a C-class, except for (3.2). Condition (3.4) is immediate, and (3.5) follows from (3.1). The lemma below gives (3.3) and (3.7).

LEMMA 3.12. *The map $\xi \to E|\xi|$ is wm-continuous and finite on $\mathfrak{D}$.*

PROOF. It suffices to show that $i(X_n) \overset{wm}{\to} \xi$ implies $E|X_n| \to E|\xi| < \infty$. Suppose first that $V_n \in H$, $E|V_n| = 1$ and $\mathcal{L}(V_n) \to \lambda$. Then by (1.2), $|\lambda| = 1$. So if $W_n \in H$ and $\mathcal{L}(W_n) \to \mu$ then by considering subsequences of $W_n/E|W_n|$ convergent in law, we see that $\sup E|W_n| < \infty$ and so by (1.2) $E|W_n| \to |\mu| < \infty$, since (W_n) is uniformly integrable. Finally, notice that $i(X_n) \overset{wm}{\to} \xi$ implies $\mathcal{L}(X_n) \to \xi^{\Omega}$, and that $|\xi^{\Omega}| = E|\xi|$.

The proof of (3.6) for $\mathfrak{D}$ is a special case of Lemma 3.14 below, since $i(X) * i(Y) = i(X + Y)$. We first quote an elementary result.

LEMMA 3.13. *Let (Y_n), (Z_n) be uniformly bounded sequences of (complex-valued) r.v.'s. Suppose $Z_n \overset{w}{\to} Z$, $Y_n \overset{w}{\to} Y$. Then there exist $k_n \to \infty$ such that $Y_n Z_{j_n} \overset{w}{\to} YZ$ whenever $j_n \to \infty$, $j_n \leqslant k_n$.*

LEMMA 3.14. *Suppose $\xi_n \overset{wm}{\to} \xi$ and $\eta_n \overset{wm}{\to} \eta$. Then there exist $j_n \to \infty$ such that $\xi_n * \eta_{j_n} \overset{wm}{\to} \xi * \eta$.*

PROOF. Recall that ϕ denotes Fourier transform. For each t we have $\phi_{\xi_n}(t) \overset{w}{\to} \phi_\xi(t)$, $\phi_{\eta_n}(t) \overset{w}{\to} \phi_\eta(t)$. By Lemma 3.13 and a diagonal argument, there exist $j_n \to \infty$ such that $\phi_{\xi_n}(t)\phi_{\eta_{j_n}}(t) \overset{w}{\to} \phi_\xi(t)\phi_\eta(t)$ for rational t. Now (2.7) gives the result.

By definition, $\mathcal{C}[H]$ is the subset of $\mathfrak{D}$ satisfying (3.2): equivalently, the subset for which $\phi_\xi(t) = \phi_\xi(-t)$ a.s. for each t. So $\mathcal{C}[H]$ is a C-class by the corresponding properties of $\mathfrak{D}$, using (2.7) to verify *wm*-closure. It only remains to prove $\mathcal{C}[H]$ is nontrivial. Because H is infinite-dimensional, we can find (X_n) in its unit ball with no s-convergent subsequence. Passing to a subsequence, assume $i(X_n) \overset{wm}{\to} \xi$. By (2.15) ξ is not degenerate, otherwise X_n is convergent in probability and hence in L^1, by (1.2). Since $\xi \in \mathfrak{D}$, (3.6) shows that $\xi * (-\xi) \in \mathfrak{D}$, in an obvious notation, and this is a nondegenerate element of $\mathcal{C}[H]$.

REMARK. Convolution of random measures does not correspond to addition of the associated exchangeable sequences. Instead, if $\xi^{(j)}$ is associated with $(Z_i^{(j)})$ then $\xi^{(1)} * \xi^{(2)}$ is associated with the exchangeable sequence which is approximated in law by $(Z_1^{(1)} + Z_{n_1}^{(2)}, Z_2^{(1)} + Z_{n_2}^{(2)}, \dots)$ for rapidly increasing (n_i).

For the proof of Proposition 3.11 we need to quote a known result. Call a sequence (Y_n) a *copy* of another sequence (Y_n') defined on a possibly different probability triple $(\Omega', \mathcal{F}', P')$ if $P((Y_1, Y_2, \dots) \in \cdot\,) = P'((Y_1', Y_2', \dots) \in \cdot\,)$.

PROPOSITION 3.15 (ALDOUS [1, THEOREM 10]; BERKES AND ROSENTHAL [3, THEOREM 1.2]). *Let $\lambda \in \mathcal{P}$ and suppose $i(X_n) \overset{wm}{\to} \lambda$. Then there exist, on some probability triple, sequences (Y_n), (Z_n) such that*

$$(Y_n) \text{ is a copy of some subsequence } (X_{j_n});$$

$$Z_1, Z_2, \dots \text{ are independent with law } \lambda$$

$$\Sigma|Y_i - Z_i| < \infty \text{ a.s.}$$

We remark that for the special case of constant α, Proposition 3.11 is a simple corollary of Proposition 3.15, because an independent sequence (Z_i) with law $\sigma(q, a)$ is isomorphic to the unit vector basis of l_q.

Proof of Proposition 3.11. For $\mathbf{a} = (a_i)$ let $\|\mathbf{a}\|_q = (\Sigma|a_i|^q)^{1/q}$ be the l_q norm. Let $c_q = |\sigma(q, 1)|$. Fix $\varepsilon > 0$. Suppose $0 < K_1 \leqslant \alpha \leqslant K_2$. We shall show that some subsequence (W_n) of (X_n) satisfies

$$K_1 c_q \|\mathbf{a}\|_q - \varepsilon\Sigma|a_i|2^{-i} \leqslant E|\Sigma a_i W_i| \leqslant K_2 c_q \|\mathbf{a}\|_q + \varepsilon\Sigma|a_i|2^{-i} \tag{3.16}$$

for all $\mathbf{a}$. First, apply Lemma 2.13 with $g(x, y) = xy$ to deduce $i(\alpha^{-1}X_n) \overset{wm}{\rightarrow} \sigma(q, 1)$. Then by Proposition 3.15 we can find a subsequence (W_n) and construct sequences (Y_n), (Z_n) such that

$$(Y_n) \text{ is a copy of } (\alpha^{-1}W_n); \tag{3.17}$$

$$Z_1, Z_2, \dots \text{ are independent, law } \sigma(q, 1); \tag{3.18}$$

$$Y_i - Z_i \to 0 \quad \text{a.s.} \tag{3.19}$$

In view of (3.17) we can construct $\hat{\alpha}$ such that $(\hat{\alpha}, Y_1, Y_2, \dots)$ is a copy of $(\alpha, \alpha^{-1}W_1, \alpha^{-1}W_2, \dots)$, and so

$$(\hat{\alpha}Y_n) \text{ is a copy of } (W_n). \tag{3.20}$$

The sequences $(\hat{\alpha}Y_n)$, $(\hat{\alpha}Z_n)$ are uniformly integrable, so (3.19) implies $E|\hat{\alpha}Y_n - \hat{\alpha}Z_n| \to 0$, and by passing to a further subsequence we may assume

$$E|\hat{\alpha}Y_i - \hat{\alpha}Z_i| \leqslant \varepsilon 2^{-i}. \tag{3.21}$$

Now $E|\Sigma a_i Z_i| = c_q\|\mathbf{a}\|_q$, and $E|\Sigma a_i W_i| = E|\hat{\alpha}\Sigma a_i Y_i|$ by (3.20), and so (3.21) and the obvious estimates give (3.16).

Next consider the case $\alpha = 0$: then $X_n \overset{s}{\rightarrow} 0$ and so some subsequence (W_n) satisfies

$$E|\Sigma a_i W_i| \leqslant \varepsilon\Sigma|a_i|2^{-i}. \tag{3.22}$$

Observe that if (3.16) or (3.22) holds for some (W_n), then it holds for any subsequence of (W_n).

To treat the general case, define

$$\begin{aligned} b_m &= 1 + m\varepsilon, && m \geqslant 0, \\ &= (1 + \varepsilon)^m, && m < 0; \\ A_m &= \{b_m < \alpha \leqslant b_{m+1}\}, \\ A_* &= \{\alpha = 0\}, \end{aligned}$$

and ignore any of these sets which are null. Put $S = \Sigma b_m P(A_m)$. Choose integers $1 < L_1 < L_2 < \dots$ such that

$$\sum_{|m| < L_1} b_m P(A_m) \geqslant S - \varepsilon, \tag{3.23}$$

$$\sup_n \sum_{|m| > L_i} E|X_n|1_{A_m} \leqslant \varepsilon 2^{-i}, \qquad i \geqslant 1. \tag{3.24}$$

Let $r_m = \min\{i\colon L_i \geqslant m\}$. It is immediate from definition (2.3) that if $\xi_n \overset{wm}{\rightarrow} \xi$ under P then $\xi_n \overset{wm}{\rightarrow} \xi$ under $P(\cdot|A)$. Thus for fixed m we can choose (W_n) such that (3.16) holds for $E(\cdot|A_m)$; and similarly (3.22) for $E(\cdot|A_*)$. Using a diagonal argument, (W_n) can be chosen such that

$$E\big(\Sigma|a_i W_i|\ |A_*\big) \leqslant \varepsilon\Sigma|a_i|2^{-i};$$

$$E(\Sigma|a_i W_i|\ |A_m) \geqslant b_m c_q\|\mathbf{a}\|_q - \varepsilon\Sigma|a_i|2^{-i}, \qquad |m| \leqslant L_1;$$

$$E\left(\left|\sum_{i>r_m} a_i W_i\right|\ |A_m\right) \leqslant b_{m+1}c_q\left(\sum_{i>r_m}|a_i|^q\right)^{1/q} + \varepsilon\sum_{i>r_m}|a_i|2^{-i}, \quad \text{each } m.$$

Combining these estimates with (3.23) and (3.24) gives

$$E|\Sigma a_i W_i| \geqslant (S-\varepsilon)c_q\|\mathbf{a}\|_q - \varepsilon\Sigma|a_i|2^{-i};$$

$$E|\Sigma\, a_i W_i| \leqslant (S+\varepsilon)c_q\|\mathbf{a}\|_q + 2\varepsilon\Sigma|a_i|2^{-i}.$$

So for ε sufficiently small, (W_i) is isomorphic to the unit vector basis of l_q.

REMARK. We made no assertion concerning the dependence between $\hat{\alpha}$ and (Z_i). If these were independent, then $(\hat{\alpha}Z_i)$ would be exchangeable. However, a modification of examples in **[1]**, [3] shows that in general we cannot achieve (3.21) with exchangeable $(\hat{\alpha}Z_i)$. In other words, we cannot directly mimic Proposition 3.15 for nonconstant random measures. Our argument depends on the special form of the random measure $\sigma(q, \alpha)$ as a randomly scaled fixed measure. Alternatively, it might be possible to reduce Proposition 3.11 to Proposition 3.15 via a "variation of density" argument used in **[10]**.

4. A fixed point theorem. Before starting the proof of Theorem 3.10, which occupies the rest of the paper, some remarks seem appropriate. As is well known, a measure $\lambda \in \mathcal{P}_0$ is of the form $\sigma(q, a)$ for some q, a if and only if there exist constants (c_n) such that $\lambda^{*n} = s_{c_n}(\lambda)$, where λ^{*n} is the n-fold convolution of λ with itself. It follows that a $\mathcal{P}_0$-valued random measure ξ is fo the form $\sigma(q, \alpha)$ if and only if $\xi^{*n} = s_{c_n}(\xi)$ for some (c_n). Now let $\mathcal{C}$ be a nontrivial C-class, and put $\mathbb{S}\mathcal{C} = \{\xi \in \mathcal{C}\colon E|\xi| = 1\}$. Define $T_n\colon \mathbb{S}\mathcal{C} \to \mathbb{S}\mathcal{C}$ by $T_n(\xi) = s_b(\xi^{*n})$, where $b = E|\xi^{*n}|$. Then Theorem 3.10 is simply the assertion that $\mathbb{S}\mathcal{C}$ contains some element fixed under the commuting family (T_n). So we can think of Theorem 3.10 as a fixed point theorem. Unfortunately there is no reasonable topology which makes $\mathbb{S}\mathcal{C}$ compact and T_n continuous (consider ξ_n as in Example 2.8). In particular, T_n is not *wm*-continuous and $\mathbb{S}\mathcal{C}$ is not *sm*-compact in general. Hence it seems impossible to reduce Theorem 3.10 to any of the standard theorems.

In the proof of Theorem 3.10 it is convenient to represent measures by their Fourier transforms. Let F be the set of functions $f\colon [0, \infty) \to [-1, 1]$ such that $f(t) = \phi_\lambda(t)$ for some $\lambda \in \mathcal{P}_0$ (we need consider only $t \geqslant 0$ because $\mathcal{P}_0$ consists of symmetric measures). Define $\|\cdot\|$ on F by $\|\phi_\lambda\| = |\lambda|$. Explicitly, this is

$$\|f\| = 4\pi^{-1}\int_0^\infty t^{-2}(1 - f(t))\, dt. \tag{4.1}$$

We now represent random measures ξ with values in $\mathcal{P}_0$ by their transforms $\phi_\xi(t)$, which we may regard as stochastic processes $X = (X_t)_{0<t}$ whose sample paths

happen to lie in F. For uniformly bounded processes $(X^n) = (X_t^n)$ we define

$$X^n \xrightarrow{wp} X \quad \text{if } X_t^n \xrightarrow{w} X_t, \qquad t \geq 0,$$

$$X^n \xrightarrow{sp} X \quad \text{if } X_t^n \xrightarrow{s} X_t, \qquad t > 0. \tag{4.2}$$

Using (2.6) and (2.7) we can translate part of §3 into transform language. A class $\mathcal{C}$ of processes X is a C-class if:

$$\text{each } X \in \mathcal{C} \text{ is } F\text{-valued and } E\|X\| < \infty, \tag{4.3}$$

$$\text{if } X^n \in \mathcal{C}, X^n \xrightarrow{wp} X \text{ then } X \in \mathcal{C} \text{ and } E\|X^n\| \to E\|X\|, \tag{4.4}$$

$$\text{if } (X_t) \in \mathcal{C} \text{ then } (X_{at}) \in \mathcal{C}, \qquad 0 \leq a < \infty, \tag{4.5}$$

$$\text{if } X, Y \in \mathcal{C} \text{ then } XY \in \mathcal{C}. \tag{4.6}$$

$\mathcal{C}$ is nontrivial if it contains some element other than $X \equiv 1$. And (3.8) gives

$$\{X \in \mathcal{C}\colon E\|X\| \leq K\} \text{ is } wp\text{-compact.} \tag{4.7}$$

Let $\mathcal{C}$ be a fixed nontrivial C-class. Theorem 3.10 is the assertion that $\mathcal{C}$ contains some element of the form $\exp(-\alpha t^q)$. Note that the subset $\mathcal{C}'$ of $\mathcal{C}$ satisfying

$$X_t \geq 0 \tag{4.8}$$

is a C-class, and is nontrivial since $X \cdot X \in \mathcal{C}'$ for $X \in \mathcal{C}$. Hence we may include (4.8) as part of the definition of a C-class.

Recall that $\mathbf{R}^+$ denotes the compactified half-line $[0, \infty]$, and $\mathcal{P}^+$ ($\mathfrak{M}^+$) the space of probability measures (random measures) on $\mathbf{R}^+$. For $X \in \mathcal{C}$ we can regard $-\log X_t$ as a $\mathbf{R}^+$-valued r.v.

The proof of Theorem 3.10 falls into two parts. First we construct elements X of $\mathcal{C}$ which are "almost" of the required form; this result, stated below, will be proved in §5.

PROPOSITION 4.9. *There exist $q \in [1, \infty]$ and elements X^j of $\mathcal{C}$ such that*
(i) $E\|X^j\| \to 1$,
(ii) $-\log X_t^j - t^q(-\log X_1^j) \xrightarrow[p]{} 0$, $t > 0$.

For definiteness, set $\infty - \infty = \infty$. Putting $\alpha_j = -\log X_1^j$, part (ii) implies $X_t^j - \exp(-\alpha_j t^q) \xrightarrow{s} 0$, $t > 0$. To establish Theorem 3.10, we must produce $X \in \mathcal{C}$ of the form $X_t = \exp(-\alpha t^q)$. The argument involves rather subtle interactions between the two topologies. We start by considering which processes can arise as wp-limits of the processes in Proposition 4.9. Fix q for the rest of this section.

DEFINITION 4.10. *Let $\mathcal{C}_q$ be the set of X such that there exist $X^n \in \mathcal{C}$ and $\mathbf{R}^+$-valued r.v.'s α_n such that*
(a) $X^n \xrightarrow{wp} X$,
(b) $X_t^n - \exp(-\alpha_n t^q) \xrightarrow{s} 0$, $t > 0$.

LEMMA 4.11. *$\mathcal{C}_q$ is a nontrivial C-class contained in $\mathcal{C}$.*

PROOF. $\mathcal{C}_q \subset \mathcal{C}$ by (4.4). For (X^j) as in Proposition 4.9, it is clear that any wp-limit X of a subsequence is an element of $\mathcal{C}_q$, and $E\|X\| = 1$. So $\mathcal{C}_q$ is nontrivial. Lemma 3.14 shows $\mathcal{C}_q$ satisfies (4.6), and the other requirements are obviously satisfied.

Call a C-class *minimal* if it is nontrivial but no proper subset is a nontrivial C-class. Since $\{X \in \mathcal{C}: E\|X\| = 1\}$ is *wp*-compact, Zorn's lemma shows that every nontrivial C-class contains a minimal subclass. Lemma 4.11 shows that if $\mathcal{C}$ is minimal then $\mathcal{C} = \mathcal{C}_q$ for some q. Thus Theorem 3.10 is reduced to the assertion below.

PROPOSITION 4.12. *Suppose* $\mathcal{C} = \mathcal{C}_q$. *Then* $\mathcal{C}$ *contains* $\exp(-\alpha t^q)$ *for some r.v.* α *with* $0 \leq \alpha < \infty$ *and* $P(\alpha = 0) < 1$.

REMARK. This of course implies $1 < q \leq 2$, which we do not assume in the argument.

PROOF. We first establish a sample path property of elements of $\mathcal{C}$. Fix $X \in \mathcal{C} = \mathcal{C}_q$, and let X^n, α_n be as in Definition 4.10. Then

$$\exp(-\alpha_n t^q) \xrightarrow{w} X_t, \qquad 0 \leq t < \infty. \tag{4.13}$$

Since $\mathfrak{M}^+$ is *wm*-compact we may suppose, by passing to a subsequence, that $i(\alpha_n) \xrightarrow{wm} \xi$, say, in $\mathfrak{M}^+$. For $t > 0$ let $f_t \in C(\mathbf{R}^+)$ be given by $f_t(x) = \exp(-t^q x)$. By (2.3), $\exp(-\alpha_n t^q) = \langle f_t, i(\alpha_n)\rangle \xrightarrow{w} \langle f_t, \xi\rangle$, and so

$$X_t = \langle f_t, \xi\rangle = \int_{[0,\infty)} \exp(-xt^q)\xi(\cdot, dx), \qquad t > 0.$$

Define Λ_q to be the set of functions f: $[0, \infty) \to [0, 1]$ such that

$$f(t) = \int_{[0,\infty)} \exp(-xt^q)\lambda(dx), \quad \text{some } \lambda \in \mathcal{P}^+. \tag{4.14}$$

We have just proved that each $X \in \mathcal{C}$ has sample paths in Λ_q. Let j: $\mathcal{P}^+ \to \Lambda_q$ be the map implicit in (4.14). Equip Λ_q with the topology τ given by

$$f_n \xrightarrow{\tau} f \quad \text{iff} \quad f_n(t) \to f(t), \qquad t > 0.$$

By standard properties of Laplace transforms,

$$j: \mathcal{P}^+ \to (\Lambda_q, \tau) \text{ is a homeomorphism.}$$

In particular, each space is compact. Define $\|\cdot\|$ on Λ_q by (4.1).

LEMMA 4.15. *Let d be any bounded metrisation of* (Λ_q, τ). *Let* X^n, X, Y *denote* Λ_q*-valued processes.*

(a) $Ed(X, Y)$ *defines a complete metric on the space of* Λ_q*-valued processes.*

(b) $Ed(X^n, Y^n) \to 0$ *iff* $X_t^n - Y_t^n \xrightarrow{s} 0$, $t > 0$.

(c) *If* $Ed(X^n, Y) \to 0$, *and if* X^n *is F-valued and* $\sup E\|X^n\| < \infty$ *then* $X^n \xrightarrow{sp} Y$.

PROOF. Parts (a) and (b) are straightforward. To proved (c) it suffices, by (b), to prove $Y_0 = 1$. Passing to a subsequence, we may suppose $d(X^n, Y) \to 0$ a.s. But $d(f_n, f) \to 0$ implies $\|f\| \leq \liminf\|f_n\|$, by (4.1) and Fatou's lemma. So another application of Fatou's lemma gives $E\|Y\| \leq \sup E\|X^n\|$. Thus $\|Y\| < \infty$ a.s., whence $Y_0 = 1$ a.s. by (4.1).

Now let ρ be the metrisation of $\mathcal{P}^+$ described in Lemma 2.16, and let d be the particular metrisation of Λ_q induced from ρ by the homeomorphism j. For this d

we can add another part to Lemma 4.15, by Lemma 2.16 and (2.7) for Laplace transforms.

(d) *If* $X^n \overset{wp}{\to} Y$ *then* $Ed(X^n, \exp(-\alpha t^q)) \to Ed(Y, \exp(-\alpha t^q))$ *for any* $\alpha \geq 0$.

Returning to the proof of Proposition 4.12, fix $\varepsilon < 1/4$ and choose $X^0 \in \mathcal{C} = \mathcal{C}_q$ with $E\|X^0\| = 1$. By Definition 4.10, (4.4) and (b) above, there exist $X^1 \in \mathcal{C}$ and $\alpha_1 \geq 0$ such that

$$|E\|X^1\| - 1| \leq \varepsilon, \quad Ed\big(X^1, \exp(-\alpha_1 t^q)\big) \leq \varepsilon.$$

By the same argument, but using (d) also, we can construct inductively $X^n \in \mathcal{C}$, $\alpha_n \geq 0$ such that

(i) $|E\|X^n\| - E\|X^{n-1}\|| \leq \varepsilon 2^{-n}$,

(ii) $Ed(X^n, \exp(-\alpha_n t^q)) \leq \varepsilon 2^{-n}$,

(iii) $Ed(X^n, \exp(-\alpha_{n-1} t^q)) \leq \varepsilon 2^{-n} + Ed(X^{n-1}, \exp(-\alpha_{n-1} t^q))$.

Putting together (ii) and (iii), $Ed(X^n, X^{n-1}) \leq 5\varepsilon 2^{-n}$. So by (a), there exists a Λ_q-valued process Y such that $Ed(X^n, Y) \to 0$. Using (i) and (c), $X^n \overset{sp}{\to} Y$. Thus $Y \in \mathcal{C}$, and by (i) and (4.4) $E\|Y\| > 0$. But from (ii) and (b),

$$\exp(-\alpha_n t^q) \overset{s}{\to} Y_t, \qquad t > 0,$$

and it follows that Y is of the form $\exp(-\alpha t^q)$.

5. The smoothing argument. Given a C-class $\mathcal{C}$ consisting of processes X, let $\hat{C} = \{-\log X \colon X \in \mathcal{C}\}$. To prove Proposition 4.9 we must produce $Y \in \hat{\mathcal{C}}$ with Y_t approximately of the form $Y_1 \cdot t^q$. We know that $\hat{\mathcal{C}}$ is closed under addition and scaling (that is, $Y \in \hat{\mathcal{C}}$ implies $Y_{at} \in \hat{\mathcal{C}}$). In the special case of a subspace H generated by independent identically distributed random variables, Proposition 3.9 gives a C-class for which $\hat{\mathcal{C}}$ consists of deterministic functions $y(t)$. In this special case, we can use Markov's fixed-point theorem to show that $\hat{\mathcal{C}}$ contains some $y(t) = at^q$: this argument is somewhat reminiscent of the proof [7] that Orlicz sequence spaces contain some l_q, which is not surprising because here H *is* an Orlicz sequence space. In the general case, we would like to associate with each $Y_t \in \hat{\mathcal{C}}$ a deterministic "average" $a(t)$, such that the map $Y \to a$ preserves addition and scaling: then we could apply the above argument to show that $a(t) \simeq a(1) \cdot t^q$ for some Y, which would almost give Proposition 4.9. Unfortunately we do not know how to produce such an average, since we have examples where $EY_t \equiv \infty$, $t > 0$. Deprived of "soft" arguments, we are forced to give an actual construction, which is tedious though elementary.

Fix $Y_t \in \hat{\mathcal{C}}$, and define an average $a(t)$ via (5.6). Choose $0 < t_1 < t_2 \ll 1$ with t_2/t_1 large, and define q by $a(t_2)/a(t_1) = (t_2/t_1)^q$. By choosing t_1, t_2 to maximize q, we find that $a(t)$ cannot be much less than $a(t_1) \cdot (t/t_1)^q$ on $[t_1, t_2]$ (Lemma 5.13). Now put $t_2/t_1 = (1 + \delta)^N$ and define a smoothed process

$$Z_t = \frac{1}{a(t_1)} \sum_{i=1}^{N} (1 + \delta)^{-iq} Y\big(t \cdot t_1 \cdot (1 + \delta)^i\big).$$

By the lower bound on $a(t)$, the extreme terms of the sum make no outstanding contribution. Since $(1 + \delta)^{-jq} Z_{t(1+\delta)^j}$ is the same sum, but taken over $j + 1 \leq i \leq j + N$, we see that Z_t is approximately $Z_1 \cdot t^q$ on an interval $[1, s]$ with $s \ll t_2/t_1$.

Finally, by choosing t_1 small we make the weights $(1 + \delta)^{-iq}/a(t_1)$ large, so they can be approximated by integers: then $Z \in \hat{\mathcal{C}}$, and we are finished. The rest of the section merely fills in the details of this construction.

PROOF OF PROPOSITION 4.9. First let us prove that we may assume each element X of $\mathcal{C}$ satisfies

$$X_t \text{ is decreasing in } t. \tag{5.1}$$

(We use "decreasing" to mean "nonincreasing".) Since the subset of $\mathcal{C}$ satisfying (5.1) is clearly a C-class, it suffices to exhibit a nontrivial X satisfying (5.1). Fix $Y \in \mathcal{C}$. By properties of Fourier transforms, $f \in F$ and $\|f\| < \infty$ imply $1 - f(t) = o(t)$ as $t \to 0$. So for $0 < \rho < 1$ and $0 < a < \infty$ the infinite product

$$Y_t^{(\rho)} = \prod_{n \geq 0} Y_{at\rho^n}$$

is sp-convergent, and hence defines an element $Y^{(\rho)}$ of $\mathcal{C}$. Choose $a = a(\rho)$ so that $E\|Y^{(\rho)}\| = 1$. Clearly $Y_{\rho t}^{(\rho)} \geq Y_t^{(\rho)}$. Now let X be a wp-limit point of $X^{(\rho)}$ as $\rho \to 1$ through the sequence $(\exp(-2^{-j}))$: it is easy to verify (5.1).

For the remainder of this section let X be some fixed process in $\mathcal{C}$ such that $E\|X\| = 1$.

Let I denote the set of continuous increasing functions g: $\mathbf{R}^+ \to \mathbf{R}^+$ such that $g(0) = 0$, $g(\infty) = \infty$. Define $Y_t = -\log(X_t)$, and consider Y as a process with sample paths in I. For $g \in I$ define

$$|||g||| = 4\pi^{-1}\int_0^\infty t^{-2}(1 - \exp(-f(t)))\, dt. \tag{5.2}$$

Then for $\mu \in \mathcal{P}_0$ we have $|||-\log \phi_\mu||| = \|\phi_\mu\| = |\mu|$. In particular, $E|||Y||| = 1$. Define θ_s: $I \to I$ by $\theta_s g(t) = g(st)$. The following properties are straightforward.

$$|||\theta_s g||| = s|||g|||, \tag{5.3}$$

$$|||g + h||| \leq |||g||| + |||h|||, \tag{5.4}$$

If $0 < |||g||| < \infty$ then the function $b \to |||bg|||$ defines a strictly increasing function in I. (5.5)

Now (5.5) can be used to define a function $a(t)$ which we shall regard as an "average" of Y_t.

DEFINITION 5.6.

$$a(0) = 0,$$

$$E\left|\left|\left|\frac{1}{a(s)}\theta_s Y\right|\right|\right| = 1, \qquad s > 0.$$

LEMMA 5.7. (i) *$a(t)$ is continuous and increasing.*
(ii) $t^{-1}a(t) \to 0$ *as* $t \to 0$.
(iii) $\limsup_{t\to 0} a(st)/a(t) < \infty$ *for each* s.
(iv) $\lim_{M\to\infty} \limsup_{t\to 0} P(Y_t/a(t) > M) = 0$.
(v) *There exist* $\eta > 0$, $\delta_3 > 0$ *such that*

$$P\left(\frac{1}{a(t)} Y_t \geq \eta\right) \geq 2\eta, \qquad t < \delta_3.$$

PROOF. Assertion (i) is an easy consequence of the definitions. To prove (ii), if $|||f||| < \infty$ then

$$|||f||| = n|||\theta_{1/n}f||| \geqslant |||n\theta_{1/n}f||| = \frac{1}{n}|||nf||| \to 0,$$

the convergence occurring because $n^{-1}(1 - \exp(-nx))\downarrow 0$. Hence $E|||n\theta_{1/n}Y||| \to 0$, and (ii) follows.

To prove (iii), define for $t < 1$

$$Y^{(t)} = \left[\frac{1}{a(t)}\right]\theta_t Y. \tag{5.8}$$

Then $Y^{(t)} = -\log X^{(t)}$, for some $X^{(t)} \in \mathcal{C}$, and from (5.4) we obtain

$$\tfrac{1}{2} \leqslant E\|X^{(t)}\| \leqslant 1. \tag{5.9}$$

We defer the proof of the next assertion.

$$\text{Let } \Gamma_k = \inf\left\{E\|X^k\|: X \in \mathcal{C}, E\|X\| \geqslant \tfrac{1}{2}\right\}. \text{ Then } \Gamma_k \to \infty. \tag{5.10}$$

Now fix $s > 1$, and choose k such that $\Gamma_k > s$. Then

$$E\left|\left|\left|\frac{k}{a(t)}\theta_t(Y)\right|\right|\right| \geqslant E|||kY^{(t)}||| = E\|(X^{(t)})^k\| \geqslant \Gamma_k \geqslant s, \qquad t < 1.$$

So for $t < s^{-1}$,

$$E\left|\left|\left|\frac{k}{a(st)}\theta_t(Y)\right|\right|\right| = s^{-1}E\left|\left|\left|\frac{k}{a(st)}\theta_{st}(Y)\right|\right|\right| \geqslant 1,$$

implying $k/a(st) \geqslant 1/a(t)$. This yields (iii).

To prove (iv), fix ε, $s > 0$. For $f \in I$, (5.2) gives the estimate:

$$\text{if } f(\varepsilon) \geqslant 1 \text{ then } |||f||| > \frac{1}{2\varepsilon}. \tag{5.11}$$

Putting $t = s\varepsilon^{-1}$,

$$\begin{aligned} P\left(\frac{1}{a(s)}Y_s \geqslant \frac{a(s\varepsilon^{-1})}{a(s)}\right) &= P\left(\frac{1}{a(t)}Y_{\varepsilon t} \geqslant 1\right) \\ &\leqslant P\left(\left|\left|\left|\frac{1}{a(t)}\theta_t(Y)\right|\right|\right| > \frac{1}{2\varepsilon}\right) \quad \text{by (5.11)} \\ &\leqslant 2\varepsilon \quad \text{by 5.6.} \end{aligned}$$

Now let $s \to 0$ and apply (iii).

Finally, suppose (v) is false. Then there exist $t_n\downarrow 0$ such that $Y_{t_n}/a(t_n) \underset{p}{\to} 0$. So, defining $Y^{(t)}$ as at (5.8), we have $Y^{(t_n)} \underset{p}{\to} 0$ and hence $X_1^{(t_n)} \overset{s}{\to} 1$. Because elements of $\mathcal{C}$ have decreasing sample paths, it follows that $X^{(t_n)} \overset{sp}{\to} 1$. Now by (4.4) $E\|X^{(t_n)}\| \to 0$, contradicting (5.9).

PROOF OF ASSERTION (5.10). We revert to regarding $\mathcal{C}$ as a set of random measures. Write $\mathfrak{D} = \{\xi \in \mathcal{C}: E|\xi| = 1\}$. By scaling it suffices to prove

$$\inf_{\mathfrak{D}} E|\xi^{*k}| \to \infty \quad \text{as } k \to \infty.$$

Defining ξ^{Ω} as in §2, we have $|\xi^{\Omega}| = E|\xi|$, and so by (3.7) and (3.8) $\{\xi^{\Omega}: \xi \in \mathfrak{D}\}$ is uniformly integrable. So there exists $b < \infty$ such that

$$\int_{|x|<b} |x| \xi^{\Omega}(dx) \geqslant \frac{1}{2}, \quad \text{each } \xi \in \mathfrak{D}.$$

Hence each $\xi \in \mathfrak{D}$ satisfies $P(\xi \in B) \geqslant 1/4b$, where

$$B = \left\{ \mu \in \mathcal{P}_0: \int_{|X|<b} |x| \mu(dx) \geqslant \frac{1}{4} \right\}.$$

It now suffices to prove $\inf_B |\mu^{*k}| \to \infty$, which is elementary. This concludes the proof of Lemma 5.7.

Consider now the quantity $q(s) = \limsup_{t \to 0} a(st)/a(t)$. Clearly $q(\cdot)$ is a submultiplicative function, so we may define

$$q = \lim_{s \to \infty} \frac{\log q(s)}{\log s}. \tag{5.12}$$

By Lemma 5.7, $q \in [1, \infty)$. We shall prove Proposition 4.9 for this value of q. Much of the work is in the next lemma, which shows that there are intervals $[t_1, t_2]$ over which the function $a(\cdot)$ almost dominates At^q. The proof involves only elementary real analysis of properties (i)–(iii) of Lemma 5.7.

LEMMA 5.13. *Given $\delta_1 > 0$ there exists $r < \infty$ such that for any $\delta_2 > 0$ there exist $0 < t_1 < t_2 < \delta_2$ such that*

(i) $\exp(1/\delta_1) \leqslant t_2/t_1 \leqslant r$,

(ii) $a(t_2)/a(t_1) = (t_2/t_1)^p$ *for some* $p \in (q - \delta_1, q + \delta_1)$,

(iii) $a(t)/a(t_1) \geqslant (t/t_1)^{p-\delta_1}$; $t_1 \leqslant t \leqslant t_2$.

PROOF. Fix $\varepsilon > 0$ and $K > \log \sup_{t \leqslant 1} a(et)/a(t)$. Define $\alpha(t) = -\log a(e^{-t})$. Then $\alpha(\cdot)$ is continuous, increasing and satisfies

$$\alpha(t) \to \infty \quad \text{as } t \to \infty, \qquad \alpha(t+1) \leqslant \alpha(t) + K,$$
$$q = \lim_{s \to \infty} s^{-1} \limsup_{t \to \infty} (\alpha(s+t) - \alpha(t)). \tag{5.14}$$

So we can choose an integer t_0 such that

$$\alpha(t+s) - \alpha(t) \leqslant s(q+\varepsilon); \qquad s, t \geqslant t_0. \tag{5.15}$$

Let $T = Kt_0\varepsilon^{-1}$. Then we can find arbitrarily large s_0 such that, putting $s_6 = s_0 + T$,

$$\alpha(s_6) - \alpha(s_0) \geqslant T(q - \varepsilon). \tag{5.16}$$

Let $\beta(\cdot)$ be the minimal concave function on $[s_0, s_6]$ such that $\beta(s_0) = \alpha(s_0)$, $\beta(s_6) = \alpha(s_6)$ and $\beta \geqslant \alpha$; the reader may find it helpful to sketch a diagram.

Put

$$s_2 = s_0 + t_0,$$
$$s_1 = \sup\{t \leqslant s_2: \alpha(t) = \beta(t)\},$$
$$s_3 = \inf\{t \geqslant s_2: \alpha(t) = \beta(t)\},$$
$$\Delta_1 = \beta'(s_1) = \beta'(s_2),$$

where β' denotes the right-hand derivative. By (5.14),

$$\alpha(s_1) - \alpha(s_0) \leqslant Kt_0. \tag{5.17}$$

Next we have

$$\Delta_1 \leqslant q + \varepsilon, \tag{5.18}$$

because

$$\begin{aligned}(s_3 - s_0)(q + \varepsilon) &\geqslant \alpha(s_3) - \alpha(s_0) \quad \text{by (5.15)}\\ &= \beta(s_3) - \beta(s_0)\\ &\geqslant \Delta_1(s_3 - s_0) \quad \text{as } \beta' \text{ is decreasing.}\end{aligned}$$

Now put

$$\begin{aligned}s_4 &= \tfrac{1}{2}(s_1 + s_6),\\ s_5 &= \inf\{t \geqslant s_4: \alpha(t) = \beta(t)\},\\ \Delta_2 &= \beta'(s_4).\end{aligned}$$

We assert

$$\Delta_2 \geqslant q - 5\varepsilon. \tag{5.19}$$

To prove this, note that

$$\begin{aligned}\beta' &\leqslant \Delta_1 \quad \text{on } (s_1, s_4),\\ &\leqslant \Delta_2 \quad \text{on } (s_4, s_6),\end{aligned}$$

and so $\alpha(s_6) - \alpha(s_1) = \beta(s_6) - \beta(s_1) \leqslant \frac{1}{2}(\Delta_1 + \Delta_2)(s_6 - s_1)$. Combine this inequality with (5.16) and (5.17), and then (5.19) follows from some algebra. It is now clear that

$$\begin{aligned}\alpha(s_5) - \alpha(s_1) &= p(s_5 - s_1), \quad \text{some } p \in [\Delta_1, \Delta_2],\\ \alpha(s) &\leqslant \alpha(s_5) + \Delta_2(s - s_5), \qquad s_1 \leqslant s \leqslant s_5,\\ T &\geqslant s_5 - s_1 \geqslant \tfrac{1}{2}(T - t_0).\end{aligned}$$

Put $t_1 = \exp(-s_5)$, $t_2 = \exp(-s_1)$. Then

$$\begin{aligned}&\exp\left(\tfrac{1}{2}(T - t_0)\right) \leqslant t_2/t_1 \leqslant \exp(T),\\ &\frac{a(t_2)}{a(t_1)} = \left(\frac{t_2}{t_1}\right)^p,\\ &\frac{a(t)}{a(t_1)} \geqslant \left(\frac{t}{t_1}\right)^{\Delta_2}, \qquad t_1 \leqslant t \leqslant t_2.\end{aligned}$$

Putting $\varepsilon = \delta_1/6$ and taking K sufficiently large, we establish Lemma 5.13: the fact that t_2 may be taken arbitrarily small arises from the fact that s_0 was taken arbitrarily large.

REMARK. Given (t_1, t_2) satisfying the conditions of Lemma 5.13, let $t_2' \in [t_2, et_2]$. Then (t_1, t_2') satisfy essentially the same conditions, though with slightly weaker constants. Thus we may add to Lemma 5.13 the requirement

(iv) *given* $0 < \theta < 1$, *we may take* $t_2/t_1 = e^{N\theta}$ *for some integer* N.

The next lemma contains the essential construction, using Lemma 5.13 and properties (iv), (v) of Lemma 5.7.

LEMMA 5.20. *Fix $K < \infty$, $0 < \theta < 1$. Then there exist $X^{(j)} \in \mathcal{C}$, $Z^{(j)}$ with sample paths in I, $p_j \to q$ and integers $m_j \to \infty$ such that*

(i) $X^{(j)} \overset{sp}{\to} 1$.

(ii) $Z_t^{(j)} + m_j \log X_t^{(j)} \underset{p}{\to} 0$, $t \geqslant 0$.

(iii) $S_j \leqslant e^{-n\theta p_j} Z^{(j)}_{e^{n\theta}} \leqslant S_j + S_j'$, $|n| \leqslant K$, *where* $S_j, S_j' \geqslant 0$ *and*

(iv) $\lim_{M\to\infty} \limsup_{j\to\infty} P(S_j' > M) = 0$.

(v) $\lim_{M\to\infty} \limsup_{j\to\infty} P(S_j > M) \geqslant \eta$, *for η as in Lemma* 5.7.

PROOF. Fix K, θ. For δ_1, δ_2 to be specified later, let t_1, t_2, p, N satisfy the conclusions (i)–(iv) of Lemma 5.13. Recall that $Y = -\log X$, where X is a fixed element of $\mathcal{C}$. Define

$$Z_s = \frac{1}{a(t_1)} \sum_{n=-K}^{N-K} \exp(-n\theta p) Y(st_1 e^{n\theta}). \tag{5.21}$$

Then Z has sample paths in I and

$$S \leqslant \exp(-n\theta p) Z(e^{n\theta}) \leqslant S + S', \qquad |n| < K, \tag{5.22}$$

where

$$S = \frac{1}{a(t_1)} \sum_{n=0}^{N-2K} \exp(-n\theta p) Y(t_1 e^{n\theta}),$$

$$S' = \frac{1}{a(t_1)} \sum_{n=-2K}^{-1} + \sum_{n=N-2K+1}^{N} \exp(-n\theta p) Y(t_1 e^{n\theta}).$$

We need an upper estimate for S' and a lower estimate for S. The first is easy, since

$$S' \leqslant \frac{2K\exp(2K\theta p)}{a(t_1)} \{ Y_{t_1} + \exp(-N\theta_p) Y_{t_2} \}$$

$$= 2K\exp(2K\theta p) \left\{ \frac{1}{a(t_1)} Y_{t_1} + \frac{1}{a(t_2)} Y_{t_2} \right\}. \tag{5.23}$$

Consider sequences $\delta_1^{(j)}, \delta_2^{(j)} \downarrow 0$, with processes $Z^{(j)}$, S_j, S_j' as above. Then (5.22) gives assertion (iii), and (5.23) and Lemma 5.7 give (iv). We now give estimates which establish (v). Recall that $\mathcal{L}$ denotes the law of a random variable.

Let η, δ_3 be as in Lemma 5.7, and put $A = \{ \mu \in \mathcal{P}^+ : \mu[\eta, \infty) > 2\eta \}$. By taking $\delta_2 \leqslant \delta_3$ we may assume $\mathcal{L}(Y_t/a(t)) \in A$, $0 < t \leqslant t_2$. The following estimate is routine.

SUBLEMMA. *If $\mathcal{L}(V_i) \in A$ and $\phi_i \geqslant 0$ then*

$$P\left(\sum \phi_i V_i \geqslant \eta^2 \sum \phi_i\right) \geqslant \eta.$$

Now the sum comprising S is in the form of the Sublemma, with

$$\phi_i = \frac{1}{a(t_1)} \exp(-n\theta p) a(t_1 e^{n\theta}).$$

Hence

$$P\left(S \geqslant \eta^2 \sum_0^{N-2K} \phi_i\right) \geqslant \eta.$$

But by Lemma 5.13(iii), $\phi_i \geqslant \exp(-i\theta\delta_1)$ and so

$$\sum_0^{N-2K} \phi_i \geqslant \frac{1-\exp(2K\theta\delta_1)\exp(-N\theta\delta_1)}{1-\exp(-\theta\delta_1)}$$

$$\geqslant \frac{1-\exp(2K\theta\delta_1 - 1)}{\theta\delta_1} \quad \text{using Lemma 5.13(i), (iv)}$$

$$\to \infty \quad \text{as } \delta_1 \downarrow 0, \text{ giving (v).}$$

To prove (i) and (ii), let $m \geqslant 1$ be integral and approximate Z by

$$Z_s' = \sum_{n=-K}^{N-K} m\left[\frac{\exp(-n\theta p)}{ma(t_1)}\right] Y(st_1 e^{n\theta}).$$

Because this sum has integer weights, $Z' = -m \log X'$ for some $X' \in \mathcal{C}$. We have the simple estimates

$$-\log X' \leqslant m^{-1}Z, \tag{5.24}$$

$$Z_t + m \log X' = Z_t - Z_t' \leqslant mNY(t \cdot t_2). \tag{5.25}$$

If we hold δ_1 fixed, then by (5.21) and Lemma 5.7(iv) we see that Z remains bounded as $\delta_2 \downarrow 0$, in the sense

$$\lim_{M\to\infty} \limsup_{\delta_2 \to 0} P(Z_t \geqslant M) = 0, \quad \text{each } t.$$

We now must produce $\delta_1^{(j)}, \delta_2^{(j)} \downarrow 0$ and $m_j \to \infty$ such that $X^{(j)}$ defined as X' above satisfies (i) and (ii). To do this, first choose δ_1 small, then choose m large so that (5.24) makes $-\log X'$ small independently of δ_2, and finally choose δ_2 small so that (5.25) makes $Z + m \log X'$ small. This establishes Lemma 5.20.

By considering Lemma 5.20 for $\theta = 2^{-i}$, $K = i2^i$, applying a diagonal argument and substituting $-m_j \log X^{(j)}$ for $Z^{(j)}$ in (iii), we can manipulate the lemma into the following form.

LEMMA 5.26. *There exist* $X^{(j)} \in \mathcal{C}, p_j \to q$ *and integers* $m_j \to \infty$ *such that*
(a) $X^{(j)} \overset{sp}{\to} 1$,
(b) $\hat{S}_j \leqslant -\exp(-n2^{-j}p_j)m_j \log X^{(j)}(e^{n2^{-j}}) \leqslant \hat{S}_j + \hat{S}_j'$, $|n| \leqslant j2^j$
where $\hat{S}_j, \hat{S}_j' \geqslant 0$ *and satisfy* (iv) *and* (v) *of Lemma* 5.20.

Now by (a), $E|||-\log X^{(j)}||| = E||X^{(j)}|| \to 0$. But by (b), $\hat{S}_j \leqslant -m_j \log X_1^{(j)}$, and it follows from (v) that $E|||-m_j \log X^{(j)}||| \to \infty$. So there exist integers n_j with $n_n/m_j \to 0$ and $E|||-n_j \log X^{(j)}||| \to 1$. Now to prove Proposition 4.9 for $(X^{(j)})^{n_j}$ it suffices to prove

$$n_j(\log X_t^{(j)} - t^q \log X_1^{(j)}) \underset{p}{\to} 0, \qquad t > 0. \tag{5.27}$$

By monotonicity, we need only consider t of the form $\exp(-n2^{-i})$. Fix t of this form. For large j, we have by (b)

$$\hat{S}_j \leqslant -m_j \log X_1^{(j)} \leqslant \hat{S}_j + \hat{S}_j',$$

$$\hat{S}_j \leqslant -t^{p_j} m_j \log X_t^{(j)} \leqslant \hat{S}_j + \hat{S}_j',$$

and (5.27) follows.

6. Miscellaneous remarks. (a) We have shown that l_q embeds into H whenever $\mathcal{C}[H]$ contains $\sigma(q, \alpha)$. The converse is true, although not quite obvious (we omit the proof). It might be hoped that, for general subspaces $H^{(i)}$ of L^1, one could attack the problem of deciding whether $H^{(1)}$ embeds into $H^{(2)}$ by looking at $\mathcal{C}[H^{(1)}]$ and $\mathcal{C}[H^{(2)}]$. But this program seems very difficult: here is one conjecture along these lines. Let $\mathfrak{M}_q$ be the *wm*-closure of $\{\sigma(q, V)\colon V \geqslant 0\}$. The usual embedding of L^q into L^1 produces a subspace H with $\mathcal{C}[H] \subset \mathfrak{M}_q$, although we can find no simple description of $\mathcal{C}[H]$. However, we conjecture that any H for which $\mathcal{C}[H]$ contains a sufficiently rich subset of $\mathfrak{M}_q$ must contain an isomorph of L^q.

(b) Given a subspace H, for which values of q does l_q embed into H? An obvious lower bound for q is $\sup\{p\colon H \text{ is type } p\}$. Our argument gives an upper bound, at least in principle: take ξ in $\mathcal{C}(H)$, put $Y_t = -\log \phi_\xi(t)$, define $a(t)$ by 5.6 and $q(\xi)$ by (5.14), and then $q \leqslant \sup\{q(\xi)\colon \xi \in \mathcal{C}(H)\}$. This is far from satisfactory. Note that even if Proposition 4.9 holds for a particular q, it does not necessarily follow that l_q embeds into H, since the argument in §4 uses a minimality assumption.

REFERENCES

1. D. J. Aldous, *Limit theorems for subsequences of arbitrarily-dependent sequences of random variables*, Z. Wahrsch. Verw. Gebiete **40** (1977), 59–82.

2. D. J. Aldous and G. K. Eagleson, *On mixing and stability of limit theorems*, Ann. Probab. **6** (1978), 325–331.

3. I. Berkes and H. P. Rosenthal, *Almost exchangeable sequences of random variables*, preprint, 1977.

4. D. Dacunha-Castelle, *Variables aléatories exchangeables et espaces d'Orlicz*, Séminaire Maurey-Schwartz 1974–1975.

5. D. Dacunha-Castelle and J. L. Krivine, *Sous-espaces de L^1*, Israel J. Math. **26** (1977), 320–351.

6. O. Kallenberg, *Random measures*, Academic Press, London, 1976.

7. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Lecture Notes in Math., vol. 338, Springer-Verlag, Berlin, 1973.

8. ______, *Classical Banach spaces*. I, Springer-Verlag, Berlin, 1977.

9. B. Maurey and G. Schectmann, *Some remarks on symmetric basic sequences in L_1*, Compositio Math. **38** (1979), 67–76.

10. H. P. Rosenthal, *On subspaces of L^p*, Ann. of Math. (2) **97** (1973), 344–373.

11. ______, *The Banach spaces $C(K)$ and $L^p(\mu)$*, Bull. Amer. Math. Soc. **81** (1975), 763–781.

12. J. L. Krivine and B. Maurey, *Espaces de Banach stables*, C. R. Acad Sci. Paris Sér. A-B **289** (1979), 679–681.

13. Séminaire d'Analyse Fonctionelle 1979–1980, Ecole Polytéchnique, Paris.

DEPARTMENT OF STATISTICS, UNIVERSITY OF CALIFORNIA, BERKELEY,